Fostering Early Number Sense

Arthur J. Baroody

University of Illinois at Urbana-Champaign

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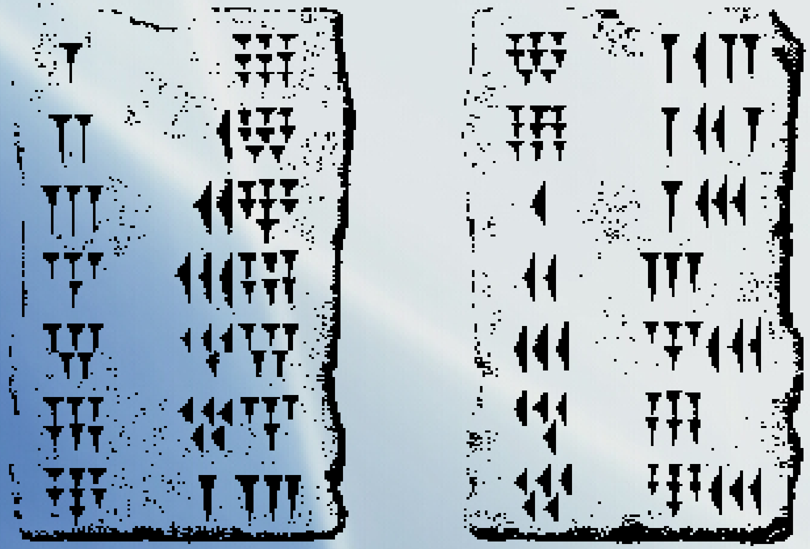
As Figure 1 illustrates, mastering (memorizing) the basic number combinations, including single-digit addition with sums to 18 (e.g., 9 + 3 = 12) and related differences (e.g., 12 – 9 = 3) has been a central goal of elementary instruction since ancient times. Yet, many pupils struggle to achieve this goal, and some fail to ever achieve it—often with terrible consequences for their progress in school mathematics and beyond.

Figure 1: Babylonian Clay Tablets of the *n* x 9 fact family (circa 4000 B.C.)



Questions Commonly Asked by Elementary (K-grade 8) Teachers and the (Perhaps Surprising) Answers Suggested by Recent Research

Question 1. *What is the best way to help students master the basic facts?*

If the goal is ***mastery with fluency***—that is, efficient (fast + accurate), appropriate, and flexible retrieval—then:

- Focus on fostering **meaningful memorization**, not memorization by rote.
- Focus on developing children's general **number sense**.

If **mastery with fluency** grows out of **meaningful memorization**, which is a by-product of a growing **number sense**, then instruction needs to **focus on helping children discover patterns and relations**.

Mastery with fluency—to paraphrase Poincare (1905)—"is built up of facts as a house is of stones, but a collection of facts is no more [fact fluency] than a pile of stones is a house" (p. 141).

Patterns and relations are the mortar that ties facts together to form well-structured knowledge.

In his 1892 “Talk to Teachers,” the eminent psychologist William James (1958) underscored the need to build on existing knowledge to promote (meaningful) memorization of knowledge:

> “…the art of remembering is the art of *thinking*; … when we wish to fix a new thing in a pupil’s [mind], our conscious effort should not be so much to *impress* and *retain* it as to *connect* it with something already there. The connecting *is* the thinking; and, if we attend clearly to the connection, the connected thing will certainly be likely to remain within recall” (pp. 101-102).

Questions Commonly Asked by Elementary (K-grade 8) Teachers and the (Perhaps Surprising) Answers Suggested by Recent Research

Question 2. What is an appropriate age to start instruction geared to foster mastery with fluency?

Early childhood;

preschool;

2 years of age!!!!!????

Why 2 and **How to?**
Or
Why Start at Such a Young Age and What Can Educators Do to Effectively Promote Number Sense and Fact Fluency?

Over the course of the twentieth century, psychologists came to

dramatically different conclusions about—

- (Issue 1) What mathematical competence can young children really achieve (i.e., what everyday or informal mathematical strengths can preschoolers be reasonably expected to realize)?

- (Issue 2) What is the basis of this informal competence (e.g., what role does language play in concept development)?

Issue 1. Changes in the Conventional Wisdom Regarding the *Nature* of Preschoolers' Informal Knowledge

Like a pendulum, the conventional wisdom about young children's informal number and arithmetic competence has swung from extremely pessimistic to extremely optimistic and then back toward the middle.

a. Early Pessimistic Views. For most of the century, psychologists held a pessimistic view and focused on what children *can't do*.

- William James (1890):
  an infant's perception of the world =
  a "great, blooming, buzzing confusion."

- <u>Edward L. Thorndike</u> (1922):

  young children are so mathematically inept that "little is gained by [doing] arithmetic before grade 2, though there are many arithmetic facts that can [be memorized by rote] in grade 1" (p. 198).

- <u>Jean Piaget</u> (1965):

  Piaget's relatively pessimistic views about young children's abilities and capacities also had the effect of limiting expectations about what young children can learn and be taught. For instance, he believed that preschoolers are—

  → *pre*operational thinkers,

  → incapable of logical and systematic thinking, or

  → incapable of constructing abstract concepts (e.g., a true concept of number or understanding of arithmetic).

Such pessimistic views of psychologists have helped reinforce a ***minimalist approach*** to early childhood mathematics education.

That is, preschool instruction often pays little attention to mathematics other than perhaps sorting and classifying, (verbal and object) counting, numeral recognition, and perhaps some arithmetic facts and shape names.

Indeed, many preschool teachers avoid math instruction all together because—

- they fear boring their students and damaging the creativity and curiosity of young children

or

- their own discomfort about mathematics.

For example, in response to surveys of their instructional practices by Juanita Copley (2004), early childhood teachers often make such comments as:

- “I don’t do mathematics.”

- “Young children shouldn’t do mathematics ... it’s not appropriate.”

b. The Shift to a Highly Optimistic View. In the last quarter of the 20th century, psychologists focused on what children *can do* (rather than what they can't do) and adopted a highly optimistic view and (Gelman, 1979).

* **Innate concept of one, two, and three** (the "intuitive numbers"). Wynn (1998) concluded that infants are born with an ability to recognize and distinguish among *oneness*, *twoness*, and *threeness.*

* **Simple arithmetic (1 + 1 = 2 & 2 – 1 = 1).** Wynn (1998) further concluded that infants and can even reason about or operate on very small numbers (e.g., recognize that one object added to another makes two and that two objects minus one is one)—all before they develop verbal-based counting competencies.

* **Innate understanding of counting.** Gelman (e.g., Gelman & Meck, 1992) argued that innate counting principles allowed infants to non-verbally count (using nonverbal tags or representations) and toddlers to quickly learn number words and how to use them to count collections.

c. The Recent Shift to Middle Ground Views. Some research over the last 15 years indicates that nativists such as Wynn (1992, 1998) may be too optimistic and that a more balanced view of young children's informal mathematical knowledge is needed.

If nativists are correct that children are born with a nonverbal and innate concept of *one* to at least *three*, then toddlers should not have difficulty with non-verbal number and arithmetic tasks that involve these "intuitive numbers."

Consider the case of 4-year-old Shytiesha.

QuickTime™ and a
H.264 decompressor
are needed to see this picture.

Task 1: "Fair Battle" (Non-Verbal Matching).

Note that Shytiesha (4 years old, at-risk)—

- correctly created a matching collection when the tester's collection consisted of only one or two items,
- but (consistently) put out four items when the tester's collection consisted of three or four items (i.e., treated collections of three and four as equivalent).

Shytiesha's performance illustrates what we have commonly found:

→ Although nativists would predict that the drop should be between 3 and 4,

young children's number performance drops off dramatically when collections are larger than 2 items.

That is, there seems to be a serious **gap between** their ability to deal with (1 or) **2 and 3** (or larger numbers).

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Task 2: "Hiding Game" (Non-Verbal Production).

The non-verbal production task, unlike the matching task, requires a child to form a mental representation of the tester's collection, because the tester hides his collection after showing it for 3 seconds.

Note that Shytiesha:

* correctly re-creates the tester's collection when it consisted of only one or two items;

* incorrectly labels the tester's collection of four items as "three" and then puts out four items;

* again puts out four items when the tester's collection was three.

For Shytiesha, the number word "three" does not appear to have the same meaning as an adult. She appears to use "three") interchangeably with ("a bunch") to mean "*many*."

Not only is there a drop off in performance between collections of 2 and 3, young children's zone of comfort is exceeded far more quickly than nativists' would predict.

Levels of number (Baroody, Benson, & Lai, 2003) or mental arithmetic competence (Dowker, 1997, 1998, 2003):

1. The ***zone of competence*** (familiarity and comfort)—can determine exact answer.
2. The ***zone of proximal (partial) competence*** (moderate familiarity and comfort)—can estimate the answer.
3. The ***zone of incompetence*** (unfamiliarity and discomfort)—"melts down" (e.g., resort to a wild guess or refuse to respond).

Two-year-old correctly creates a matching collection by putting out 2.

Shown 4, the child puts out all items available—“a bunch” (melts down)

Asked to match 4 again, child pushes dinosaurs away…

And then pushes his mat into the tester’s mat.

Tester shows 2-year-old child 2 dinosaurs.

Tester hides 2 dinosaurs (Hiding Game).

Child puts out <u>one</u>; *two* appears to be in the child's zone of partial competence.

Tester shows and then hides 4 dinosaurs.

Child leaves (melts down); *four* is in the child's zone of incompetence.

Implications

* A basic understanding of number, including the concepts of *two* and *three*, are probably not innate but must be constructed.

* Without a concept of number, a concept of arithmetic is not possible, and without an understanding of arithmetic, it makes little sense to teach basic arithmetic facts.

* Social intervention is needed to help children construct an understanding of number, including the concepts of *two* and *three*.

* Fostering number sense, including an understanding of *two* and *three*, cannot be imposed or rushed. *How then can social intervention foster such knowledge?*

Issue 2. Changes in the Conventional Wisdom About the *Basis* of Preschoolers' Informal Knowledge of Number and Arithmetic

Like a pendulum, the conventional wisdom regarding the role of language in number development has also shifted back and forth over the last 100 years or so (see Mix, Sandhofer, & Baroody, 2005, for a detailed discussion).

a. Early Counting-Based View. Dewey (1898) and Thorndike (1922) concluded that children's initial training in mathematics should focus on counting.

b. Nonverbal Number Concept Before a Verbal-Based Concept View. For most of the 20^{th} century, psychologists have believed that a number concept develops independently of counting and before children acquire an understanding of verbal numbers.

- Piaget (1965), for example, dismissed verbal and object counting as skills learned by rote, skills that had no impact on constructing a number concept. He argued that the construction of a number concept depended on the development and synthesis of the logical thinking abilities necessary for classifying and ordering.

- Nativists have argued that children have an innate non-verbal concept of at least the intuitive numbers and innate (nonverbal) counting processes.

c. Language in the Form of the First Few Number Words Plays a Key Role in the Construction of Number, Counting, and Arithmetic Concepts and Skills.

* *Concept of Cardinal Number*

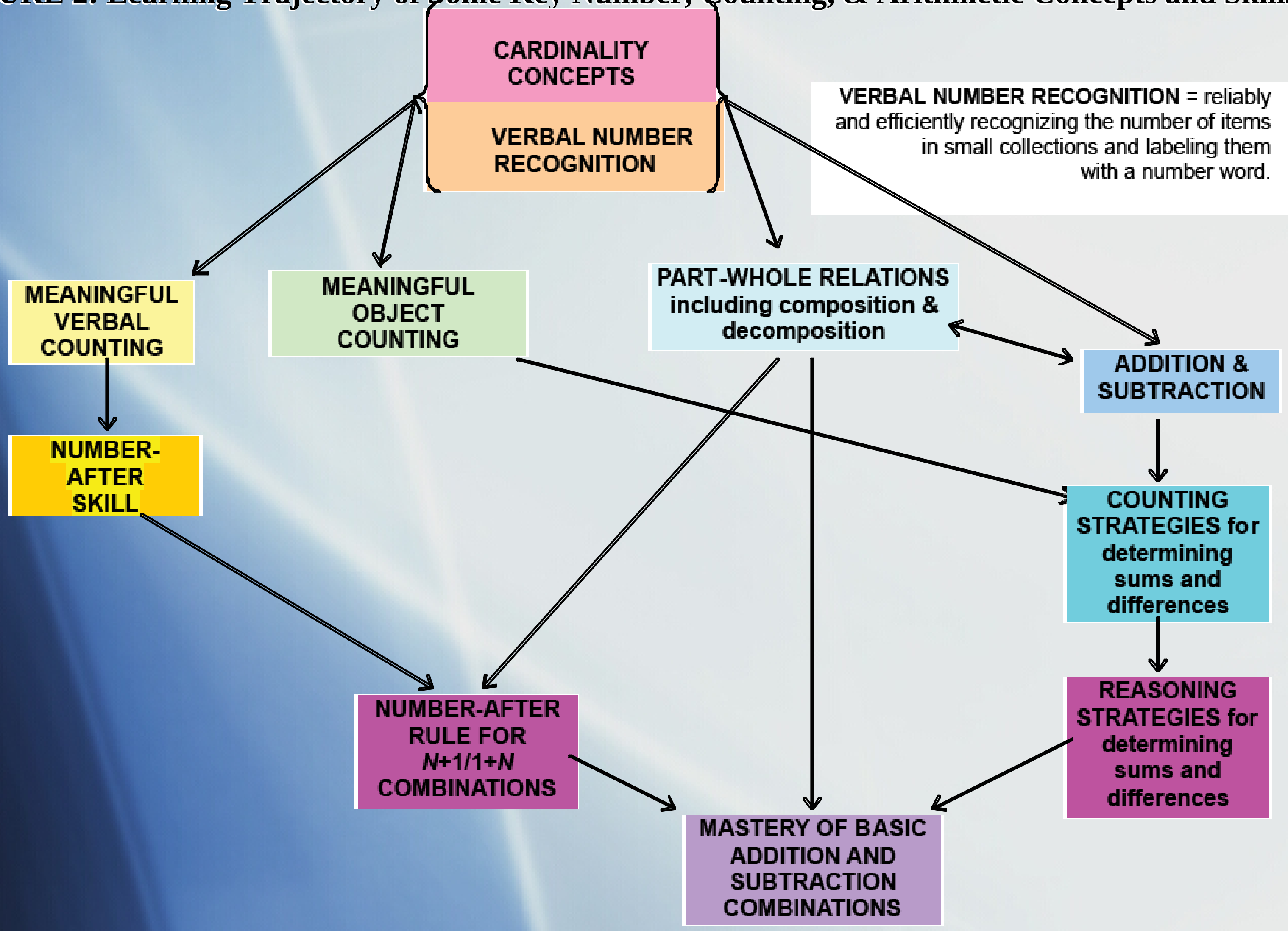
The use of "one," "two," "three" in conjunction with seeing examples and non-examples of each can help children construct an increasingly reliable and accurate number concept of one, two, and three—an understanding of oneness, twoness, and threeness.

- → By seeing ••, ΔΔ, and º o (examples of pairs), e.g., all labeled "two," young children can recognize that the appearance of the items in the collections is not important (shape and color are irrelevant to number).
- → Seeing •, •••, Δ, ΔΔΔ, , and and (non-examples of pairs) labeled as "not two" or with another number word can help them define the boundaries of the concept of two.

* *Skill of Verbal Number Recognition*

= reliably and efficiently recognizing the number of items in small collections and labeling them with the appropriate number word.

FIGURE 2: Learning Trajectory of Some Key Number, Counting, & Arithmetic Concepts and Skills



- ***Meaningful Verbal Counting***

Verbal number recognition of successively larger collections

↓

* ***Stable order principle*** (Gelman & Gallistel, 1978)**—**the order of number words order matters; number words are recited in the same order:

"one, two, three…"

*** *Ordinal meaning of number*—**numbers (number words) represent collections of different sizes;

number word sequence ("one, two, three…") represents increasingly larger collections.

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Number-after skill—the ability to start at any point in the counting sequence and (efficiently) state the next number word in the sequence.

Number-After Skill

↓

Reasoning Strategy for n+1/1+n combinations

The INSIGHT that facts involving one are related to their (already efficient) number-after knowledge— permits children to construct the number-after rule:

"The sum of n+1 or 1+ n is the number after n in the counting sequence."

→ Allows children to <u>efficiently deduce the sum of any *n*+1 fact</u> for which they know the counting sequence, even those NOT previously PRACTICED including large multidigit facts such as 28 + 1, 128 + 1, or 1,000,128+1.

→ Repeated practice of the *n*+1/1+*n* facts is unnecessary for mastery.

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Mastering n+1 and 1+n Combinations With Fluency:

With practice, the number-after rule for *n*+1/1+*n* facts becomes automatic—
can be applied quickly, efficiently, and without deliberation
(i.e., becomes a component in the retrieval network).

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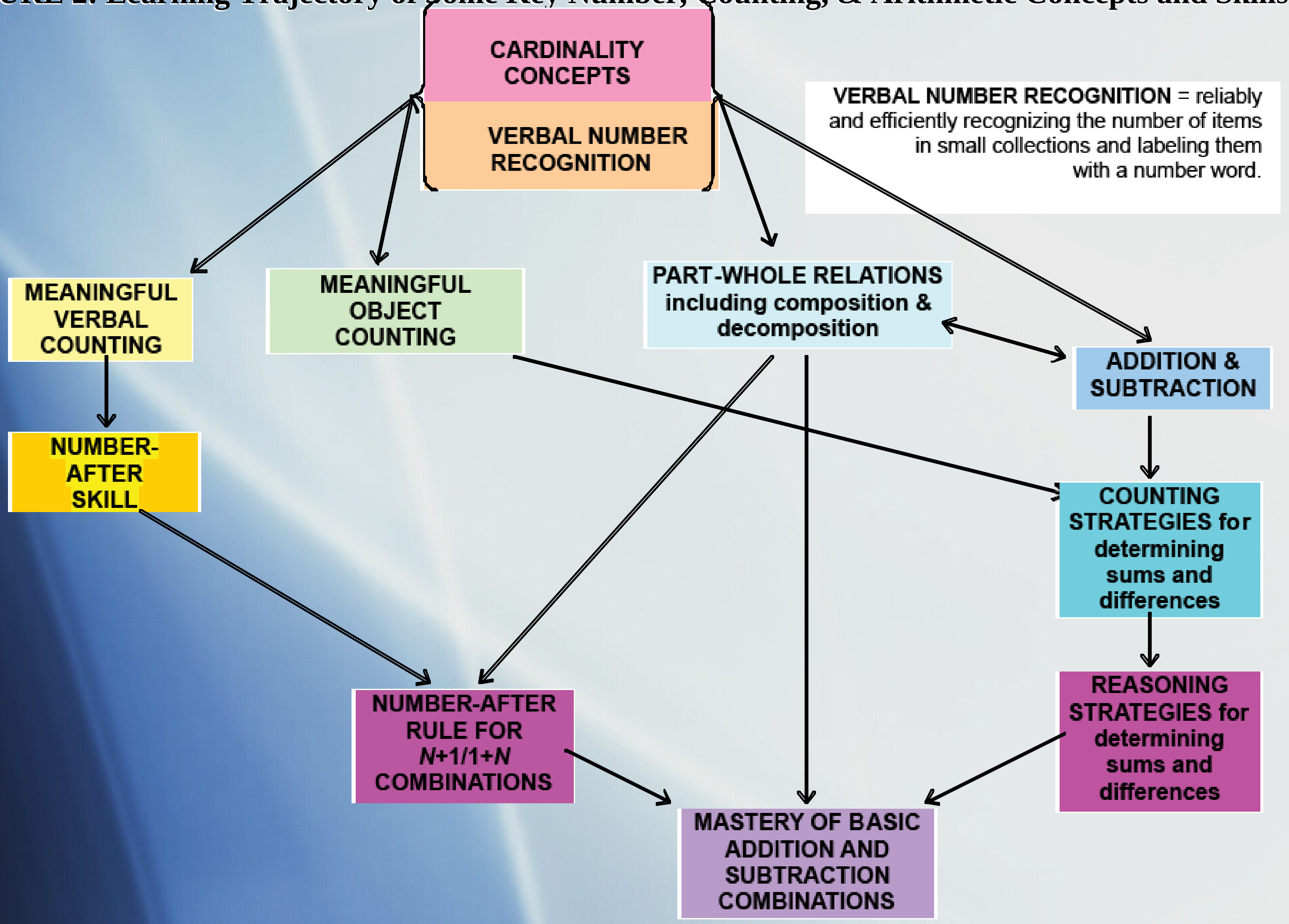

- ***<u>Meaningful Object Counting</u>***

  Verbal number recognition can enable children to understand the why and the how of **object counting**.

  → The common practice of beginning math instruction with verbal and objects counting can be difficult and confusing to children.

  → Verbal number recognition in conjunction with seeing enumeration modeled by others can enable children to discern—

  * the purpose of object counting (its another way of determining the total number of items of a collection or its cardinal value)

  and

  * the rationale for object-counting procedures such as why others emphasize or repeat the last number word used in the counting process (it represents the total number of items or cardinal value of the collection).

Key instructional implication:

Help children develop verbal number recognition by labeling examples and non-examples before trying to teach children object counting.

Meaningful Object Counting

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Using ***counting strategies*** (with objects or number words) to determine sums and differences.

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Using ***reasoning strategies*** (known facts and relations) to deduce the answer of an unknown combination.

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Using ***retrieval strategies***—efficiently producing answers from a memory network.

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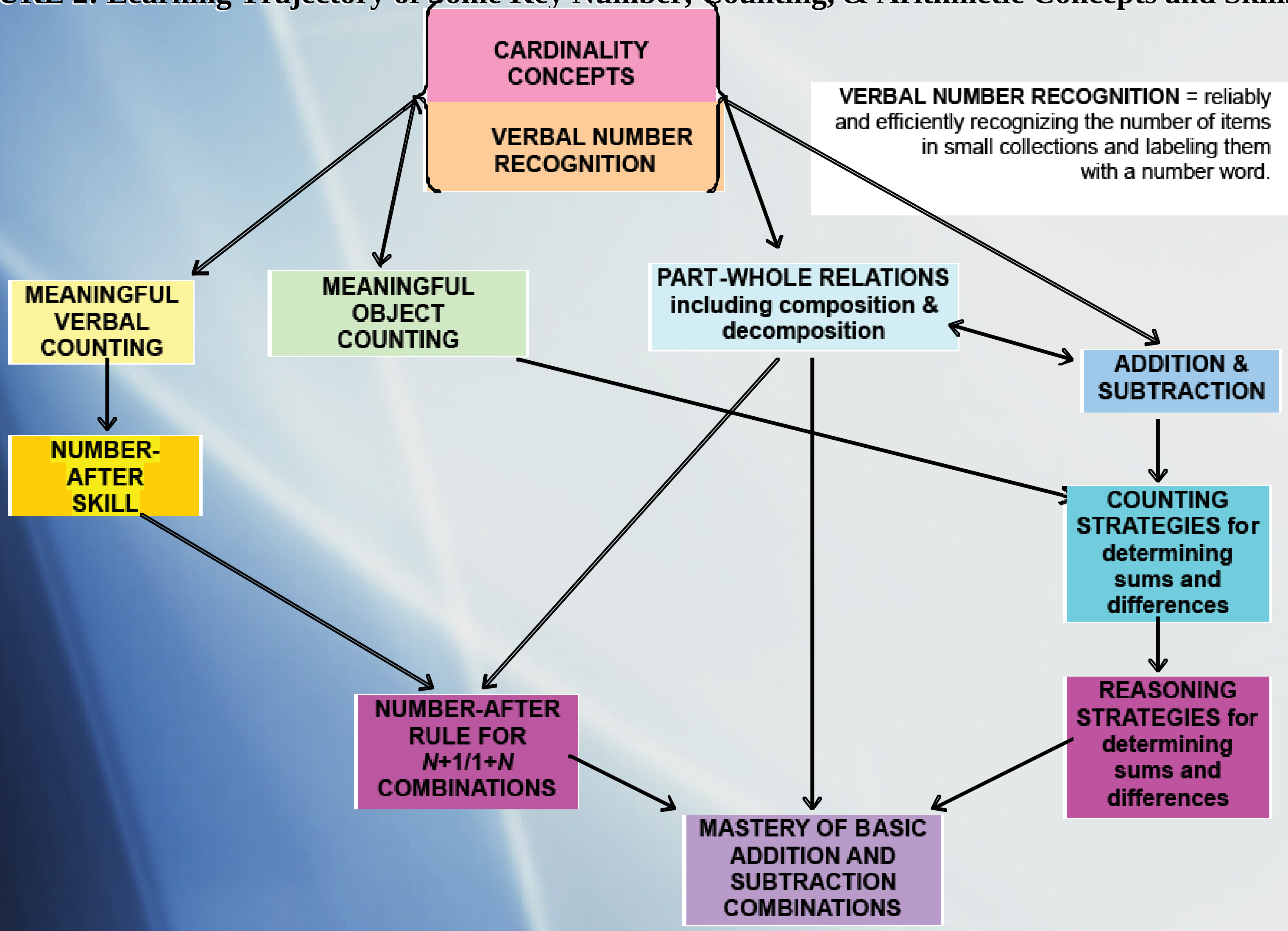

- *Concept of Composition & Decomposition.* Verbal number recognition enables children to see—
  - → one & one as *two*;
  - → one & one & one as *three*; or
  - → two & one as *three*.

The case of Alice (Baroody & Rosu, 2006):

At 25 months of age, Alice observed: "A big stone! A small stone. Another small stone. Two small stones!"

At 30 months, Alice pointed to a picture in the book her mother was reading and said, "A girl, a boy, another girl...three!"

At 28 months, asked how many crayons she had, Alice answered, "Two [raising two crayons] and one" [raising one crayon with the other hand]; "three!"

Such experiences provide the basis for an understanding of **composition**—

- → that *a collection (whole) can be built up from units (individual parts)* and
- → that *a whole often can be composed from parts in different ways*.

Verbal number recognition also enables children to see—

→ *two* as one & one;

→ *three* as one & one & one; and

→ *three* as two & one; and

→ even larger numbers as composites of one and two.

At 25 months of age, Alice was shown a picture of five soldiers and observed, *“Two, two, and one!”*

Such experiences provide the basis for understanding **decomposition**—

→ that a collection (whole) can be broken down into units (individual parts) and

→ that a whole can often be decomposed in different ways.

Verbal Number Recognition

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Repeatedly seeing the composition and decomposition of *two* and *three*
(e.g., “two” as “one and one”)*.*

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Mastering the simplest addition and subtraction combinations with fluency:
“one and one is two”;
“two and one is three”;
“two take away one is one”;
“three take away one is two”;
“three take away two is one

At 28 months of age, Alice was asked by her mother how many crayons she had. The girl answered, “Two [raising two crayons] and one” [raising one crayon with the other hand]; “three!”

Verbal Number Recognition

↓

Repeatedly seeing the composition and decomposition of *numbers up to about five*

+

<u>Feedback</u>

↓

Mastering the sums up to five and related subtraction combinations with fluency:

<u>At 30 months of age,</u> *shown a picture of four puppies, Alice put two fingers of her left hand on two dogs and said, "Two." While maintaining this posture, she placed two fingers of her right hand on the other two puppies and said, "Two." She then used the known relation "2 & 2 makes 4" (learned from her parents) to specify the cardinal value of the collection.*

Verbal Number Recognition

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Repeatedly seeing the composition and decomposition of *numbers up to about four*

(with or without feedback)

↓

number-after rule: "The sum of n+1 is the number after n in the counting sequence")

→ This rule allows children to efficiently deduce the sum of any *n*+1 fact for which they know the counting sequence, even those NOT previously PRACTICED including large facts such as 1,000,128+1 (Baroody, 1989b, 1992, 1995).

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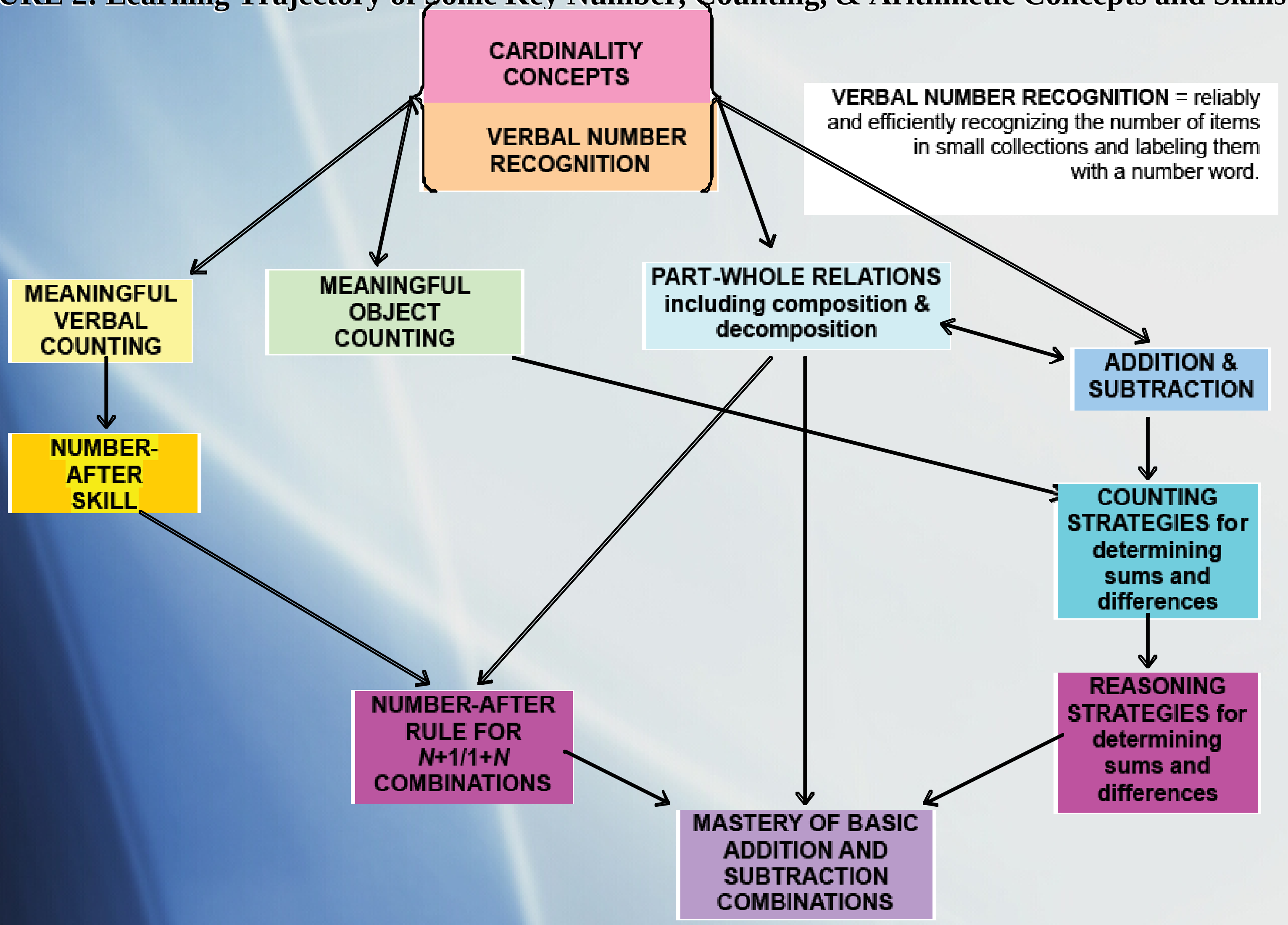
Mastering n+1 and 1+n Combinations
With Fluency

FIGURE 2: Learning Trajectory of Some Key Number, Counting, & Arithmetic Concepts and Skills



- **Basic *Addition* and Subtraction Concepts**
  Verbal number recognition (and a concept of composition and decomposition) can also provides the basis for constructing a basic understanding of—

  → **Concept of addition and subtraction.** By adding an item to a collection of two items, for example, a child can literally see that the original collection has been transformed into a larger collection of *three* or vice versa.

  → **Concept of subtractive negation.** For example, by recognizing that two blocks take away two blocks leaves none, children may induce the pattern that *any number take away itself leaves nothing*.

↓

Meaningful mastery of the $n - n = 0$ family of facts

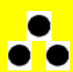
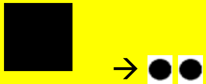
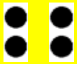
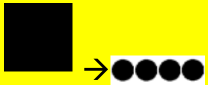
→ **Concept of additive and subtractive identity.** For example, by recognizing that two blocks take away none leaves two blocks, children may induce the regularity that *a number take way none leaves the number unchanged*.

↓

Meaningful mastery the $n - 0 = n$ family of facts

Study of 3 to 7 Year Olds' Understanding of Subtraction

(Baroody, Lai, Li, & Baroody, in press)

	Relatively Concrete Trials				Relatively Abstract Trials	
	Starting amount	Starting amount covered	Subtraction	Outcome?		
Subtrac-tion					Two million cookies	Take away a million
Negation					A million	Take away a million
Identity			Take away none 		A gazillion	Take away none

Principle	Age (years)	Relatively Concrete Trials: Marginally successful	Relatively Concrete Trials: Reliably successful		Relatively Abstract Trials: Marginally successful	Relatively Abstract Trials: Reliably successful
Negation	3	12.5%	31.3%		6.3%	18.8%
	4	6.3%	87.5%		6.3%	75%
	5	0%	100%		0%	93.8%
Identity	3	25%	50%		25%	6.3%
	4	0%	87.5%		12.5%	62.5%
	5	0%	100%		6.3%	87.5%

Note. Success on the negation and identity tasks was strongly related to the ability to recognize one and two.

Conclusion:

To ensure children are developmentally ready to master the basic addition and subtraction combinations—

- Start early

and

- Focus first on helping them achieve verbal number recognition of the intuitive numbers.